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# A GENERALIZATION OF LE POTIER'S VANISHING THEOREM

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## 1. INTRODUCTION

Consider a vector bundle  $E$  of rank  $d$  over a compact complex manifold  $X$  of dimension  $n$ , and a partition  $R = (r_1, r_2, \dots, r_m)$  of weight  $r = \sum_{i=1}^m r_i = |R|$ , where the  $r_i$  are strictly positive integers with  $r_i \geq r_{i+1}$  and  $m \leq d$ . We call  $m$  the length of  $R$ .

Let  $\wedge_R = \mathcal{S}_{\tilde{R}}$  be the Schur functor (for the definition see [M, p.45]) corresponding to the transpose  $\tilde{R}$  of  $R$ . Schur functors were initially defined on the category of vector spaces and linear maps, but by functoriality the definition carries over to vector bundles on  $X$ .

We prove the following vanishing theorems for cohomology groups:

**Theorem 1.1.** *For any partition  $R = (r_1, r_2, \dots, r_m)$ ,*

$$H^{p,q}(X, \wedge_R E) = 0 \quad \text{if} \quad \wedge_R E \text{ is ample} \quad \text{and} \quad p+q-n > \sum_{i=1}^m r_i(d-r_i).$$

If  $m = 1$  we get Le Potier vanishing theorem [LP1](see section 2).

**Corollary 1.2.**  $H^{p,q}(X, \otimes_{i=1}^l \mathcal{S}^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E) = 0$

*if  $\otimes_{i=1}^l \mathcal{S}^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E$  is ample and*

$$p+q-n > \sum_{j=1}^m s_j(d-s_j) + (d-1) \sum_{i=1}^l k_i.$$

In [LP2] Le Potier introduced a very useful tool for the derivation of such theorems. It is based on the Borel-Le Potier spectral sequence, a term introduced by Demailly [D1].

Given a sequence of integers  $0 = s_0 < s_1 < s_2 < \dots < s_l \leq d$ , and a complex vector space  $V$  of dimension  $d$ , let  $\mathcal{F}l_{s_1, \dots, s_l}(V) = \mathcal{F}l_s(V)$  be the variety of partial flags

$$V_{s_l} \subset V_{s_{l-1}} \subset \dots \subset V_{s_1} \subset V, \quad \text{codim } V_{s_i} = s_i.$$

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This manifold carries canonical vector bundles  $Q_i$  with fibers  $V_{s_{i-1}}/V_{s_i}$ . Let  $Y = \mathcal{F}l_s(E)$  be the natural fibered variety with projection  $\pi : Y \rightarrow X$  and fibers  $\mathcal{F}l_s(E_x)$ ,  $x \in X$ .

We also denote by  $Q_i$  the corresponding vector bundle over  $Y$ . For partitions  $\Lambda_i$ , a vector bundle of the form  $\otimes_i \mathcal{S}_{\Lambda_i}(Q_i)$  over  $Y$  will be called of Schur type.

The projection  $\pi$  yields a filtration of the bundle  $\Omega_Y^P$  of exterior differential forms of degree  $P$  on  $Y$ , namely

$$F^p(\Omega_Y^P) = \pi^* \Omega_X^p \wedge \Omega_Y^{P-p}.$$

The corresponding graded bundle is given by

$$F^p(\Omega_Y^P)/F^{p+1}(\Omega_Y^P) = \pi^* \Omega_X^p \otimes \Omega_{Y/X}^{P-p},$$

where  $\Omega_{Y/X}^{P-p}$  is the bundle of relative differential forms of degree  $P - p$ . For a given line bundle  $\mathcal{L}$  over  $Y$ , the filtration on  $\Omega_Y^P$  induces a filtration on  $\Omega_Y^P \otimes \mathcal{L}$ . This latter filtration yields the Borel-Le Potier spectral sequence, which abuts to  $H^{P,q}(Y, \mathcal{L})$ .

It is given by the data  $X, Y, \mathcal{L}, P$  and will be denoted by  ${}^P\mathcal{E}_B$ . Its  $\mathcal{E}_1$ -terms

$${}^P\mathcal{E}_{1,B}^{p,q-p} = H^q(Y, \pi^*(\Omega_X^p) \otimes \Omega_{Y/X}^{P-p} \otimes \mathcal{L})$$

can be calculated as limit groups of the Leray spectral sequence  ${}^{p,P}\mathcal{E}_L$  associated to the projection  $\pi$ , for which

$${}^{p,P}\mathcal{E}_{2,L}^{q-j,j} = H^{p,q-j}(X, R^j \pi_* (\Omega_{Y/X}^{P-p} \otimes \mathcal{L})).$$

For a suitably chosen ample line bundle  $\mathcal{L}$  (see section 6) of Schur type and for  $P - p = 0$  one obtains

$${}^P\mathcal{E}_{1,B}^{P,q-P} = H^{P,q}(X, \wedge_R E).$$

Moreover, under the condition

$$(*) \quad P + q > n + \sum_{i=1}^m r_i(d - r_i),$$

the corresponding Borel-le Potier spectral sequence will be shown to degenerate at  ${}^P\mathcal{E}_{1,B}^{P,q-P}$ , in the sense that all  ${}^P d_{i,B}$  mapping to and from  ${}^P\mathcal{E}_{i,B}^{P,q-P}$  are zero, such that  $H^{P,q}(X, \wedge_R E)$  is a subquotient of  $H^{P,q}(Y, \mathcal{L})$ . The latter group vanishes by the Kodaira-Akizuki-Nakano vanishing theorem.

The map  ${}^P d_{i,B}$  from  ${}^P\mathcal{E}_{i,B}^{P,q-P}$  is zero by construction, since  $F^p(\Omega_Y^P) = 0$  for  $p > P$ . For the maps to  ${}^P\mathcal{E}_{i,B}^{P,q-P}$  we shall use an induction argument to show that their sources vanish under the condition (\*).

Because of the Leray spectral sequence, these sources are given by subquotients of groups of type  $\oplus_j H^{p,q-j}(X, R^j\pi_*(\Omega_{Y/X}^{P-p} \otimes \mathcal{L}))$ .

Since  $\Omega_{Y/X}^{P-p} \otimes \mathcal{L}$  is of Schur type, the vector bundle  $R^j\pi_*(\Omega_{Y/X}^{P-p} \otimes \mathcal{L})$  on  $X$  is given by a Schur functor applied to  $E$ . This Schur functor can be calculated for the case where  $X$  is a point and  $E$  a vector space  $V$ . Thus the core of our proof will be the evaluation of the groups  $H^{P-p,j}(\mathcal{F}l_s(V), \mathcal{L})$ . For that, we need a number of technical preparations.

In section 3 we explain some basic tools, in particular concerning Young diagrams, which label the Schur functors. In section 4 we reformulate the Littlewood-Richardson rules for the tensor product of two Schur functors. This goes slightly beyond what we need for the proof, but should have independent interest. In section 5 we consider the relevant cohomology groups on partial flag varieties, in particular on the Grassmannians. Section 6 contains the proof of the main theorem.

## 2. SOME KNOWN RESULTS

Vanishing theorems for ample vector bundles play a crucial role in algebraic geometry and its applications.

Let us recall the most important ones.

**Theorem 2.1** (Kodaira - Akizuki - Nakano ). [KAN]

*Let  $L$  be an ample line bundle on an  $n$ -dimensional projective manifold  $X$ .*

*Then  $H^{p,q}(X, L) = 0$  for  $p + q > n$ .*

The special case  $p = n$  is due to Kodaira. In this case the ampleness condition can be relaxed to yield the Kawamata-Viehweg vanishing theorem [KV].

The extension of these results to vector bundles of higher rank is due essentially to Le Potier.

In the sequel,  $E$  is a vector bundle of rank  $d$  on a  $n$ -dimensional compact complex manifold  $X$ .

**Theorem 2.2** (Le Potier). [LP2]

*If  $E$  is ample, then  $H^{n,q}(X, \wedge^r E) = 0$  for  $q > d - r$ .*

**Theorem 2.3** (Le Potier). [LP1]

*If  $\wedge^r E$  is ample, then  $H^{p,q}(X, \wedge^r E) = 0$  for  $p + q - n > r(d - r)$ .*

Although Le Potier states this theorem under the hypothesis that  $E$  is ample, his proof requires the weaker hypothesis that  $\wedge^r E$  be ample.

Indeed, in his proof he needs  $\det Q$  on  $G_r(E)$  to be ample but  $\det Q = \mathcal{O}_{\mathbb{P}(\wedge^r E)}(1)|_{G_r(E)}$ . Thus  $\wedge^r E$  ample implies that  $\det Q$  is ample.

**Theorem 2.4** (Sommese). [S]

*Let  $E_j, 1 \leq j \leq m$  be ample vector bundles of rank  $d_j$ . Then*

$$H^{p,q}(X, \otimes_{j=1}^m \wedge^{s_j} E_j) = 0 \quad \text{if} \quad p + q - n > \sum_{j=1}^m s_j(d_j - s_j).$$

Note that Corollary 1.2 gives Sommese's vanishing theorem with  $E_j = E$  for all  $j$ , under weaker assumption.

For vectors bundles tensored with powers of  $\det E$ , one obtains less restrictive vanishing conditions. The best known case is

**Theorem 2.5** (Griffiths). [G]

*Let  $E$  be ample. Then  $H^{n,q}(X, S^r E \otimes \det E) = 0$  for  $q > 0$ .*

The finite dimensional irreducible representations of  $Gl(V)$ , where  $V$  is vector space of dimension  $d$  are in correspondence with partitions of length at most  $d$ . We denote the irreducible  $Gl(V)$ -module corresponding to the partition  $R$  by  $S_R(V)$ , and call it the Schur functor of  $V$ .

In particular  $S^k V = \mathcal{S}_{(k)} V$ , and  $\wedge^h V = \mathcal{S}_{\underbrace{(1, 1, \dots, 1)}_{h \text{ times}}} V$ .

An extension of the last theorem to Schur functors is

**Theorem 2.6** (Demailly). [D1]

*Let  $R = (r_1, r_2, \dots, r_m)$  be any partition of length  $m$ . If  $E$  is ample, then  $H^{n,q}(X, \mathcal{S}_R E \otimes (\det E)^m) = 0$  for  $q > 0$ .*

Note that this theorem can be derived from that of Griffiths, as follows:

Let  $R = (r_1, \dots, r_m)$  be a partition of length  $m$ , and  $r = r_1 + \dots + r_m$  the weight of  $R$ . For  $V = \underbrace{E \oplus E \oplus \dots \oplus E}_{m \text{ times}}$ , we have

$$\mathcal{S}_R E \otimes (\det E)^m \subset S^{r_1} E \otimes S^{r_2} E \otimes \dots \otimes S^{r_m} E \otimes (\det E)^m \subset S^r V \otimes \det V.$$

Then we use Theorem 2.5.

An extension of the last result to the whole Dolbeault cohomology is due to Manivel [M2].

For  $E$  ample and arbitrary partition  $\lambda$  the question of finding an exact condition for  $H^{p,q}(X, \mathcal{S}_\lambda E)$  to vanish is still open. For a precise conjecture in the case  $p = n$  see [L].

In [LN], as a special case of a more general result, this conjecture is proved for any  $(p, q)$  and any hook Schur functor  $\Gamma_k^\alpha E$ .

The latter are defined for  $0 \leq \alpha < k$  and correspond to the partition  $(\alpha + 1, 1, \dots, 1)$  of weight  $k \in \mathbb{N}^*$ . Inductively, they can be defined as

follows:

$$\Gamma_k^0 E = \wedge^k E$$

and

$$\wedge^{k-\alpha} E \otimes S^\alpha E = \Gamma_k^{\alpha-1} E \oplus \Gamma_k^\alpha E$$

for  $0 < \alpha < k$ . In particular,  $\Gamma_k^{k-1} E = S^k E$ . Note that  $\Gamma_k^\alpha E = 0$  for  $d - k + \alpha < 0$ .

Define a function  $\delta : \mathbb{N} \rightarrow \mathbb{N}^*$  by:

$$\binom{\delta(x)}{2} \leq x < \binom{\delta(x)+1}{2}.$$

In other words,

$$\begin{aligned} \delta(0) &= 1. \\ \delta(1) &= \delta(2) = 2 \\ \delta(3) &= \delta(4) = \delta(5) = 3 \\ \delta(6) &= \delta(7) = \delta(8) = \delta(9) = 4 \end{aligned}$$

etc...

**Theorem 2.7.** *If  $E$  is ample, then  $H^{p,q}(X, \Gamma_k^\alpha E) = 0$  for  $q + p - n > (\delta(n - p) + \alpha)(d - k + 2\alpha) - \alpha(\alpha + 1)$ .*

For  $\alpha = k - d$  one obtains  $H^{p,q}(X, S^\alpha E \otimes \det E) = 0$ , when  $p + q - n > (\delta(n - p) - 1)(k - d)$ . For  $p = n$ , this specializes to Griffiths' vanishing theorem.

In the present paper we prove an extension to Schur functors of Le Potier's theorem (Theorem 2.3).

### 3. BASICS DEFINITIONS AND TOOLS

#### 3.1. Some notations and definitions.

$$\mathbb{N}^* = \mathbb{N} - \{0\}, \quad I(r) = \{1, 2, \dots, r\} \subset \mathbb{N}^*.$$

For  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ , we call  $i$  the height and  $j$  the width of  $(i, j)$ . For  $S \subset I(r) \times \mathbb{Z}$ ,  $\text{card}(S) < \infty$ , we define the sequence

$$[S] : I(r) \longrightarrow \mathbb{N} \text{ by } [S]_i = \text{card}\{j \in \mathbb{Z} \mid (i, j) \in S\}.$$

For  $S \subset I(r) \times \mathbb{N}^*$ ,  $\text{card}(S) < \infty$ , we define

$$\langle S \rangle : \mathbb{N}^* \rightarrow \mathbb{N} \text{ by } \langle S \rangle_j = \text{card}\{i \in I(r) \mid (i, j) \in S\}.$$

Partitions  $u$  of length  $l(u) \leq r$  are weakly decreasing sequences  $u : I(r) \rightarrow \mathbb{N}$ . More precisely, we regard partitions  $u = (u_1, u_2, \dots, u_r)$  and  $(u_1, u_2, \dots, u_r, 0, \dots, 0)$  as equivalent. For  $u_r \neq 0$  the length of  $u$  is  $r$ .

The weight  $u$  is given by

$$|u| = \text{card}(Y(u))$$

where

$$Y(u) = \{(i, j) \in I(r) \times \mathbb{N}^* \mid 1 \leq j \leq u_i\}$$

is the Young diagram of  $u$ . Equivalently,  $|u| = \sum_i u_i$ .

For example the Young diagram of  $u = (4, 2, 1, 0)$  is

$$Y(u) = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1)\},$$

the length is 3 and the weight is  $|u| = 7$ .

The transpose  $\tilde{u}$  of a partition  $u$  is defined by

$$Y(\tilde{u}) = \widetilde{Y(u)},$$

where  $\widetilde{(i, j)} = (j, i)$  for  $(i, j) \in \mathbb{N}^* \times \mathbb{N}^*$ . For non positive integers  $j$  we put  $\tilde{u}_j = +\infty$ .

We define the squared norm of a partition  $u$  by  $\|u\|^2 = \sum_{i \in \mathbb{N}^*} \tilde{u}_i^2$ .

When  $\psi$  is a map  $I(r) \rightarrow \mathbb{Z}$ , we denote the corresponding weakly decreasing sequence by  $\psi^{\geq}$ . More precisely,

$$\psi^{\geq} = \psi \circ \sigma,$$

where  $\sigma$  is any permutation of  $I(r)$  such that  $\psi \circ \sigma$  is weakly decreasing. When the image of  $\psi$  lies in  $\mathbb{N}^*$ ,  $\psi^{\geq}$  is a partition of length  $r$ .

When  $u$  is a weakly decreasing sequence, let  $u^>$  be the sequence obtained by removing repetitions, in other words the strictly decreasing sequence which has the same set of terms as  $u$ . For a finite sequence  $u$  let  $u^<$  be the strictly increasing sequence which has the same set of terms as  $u$ .

Every partition  $u$  can be reconstructed from  $a = u^>$  and  $s = \tilde{u}^<$ . We write  $u = a_s$ . Explicitly

$$a_s = (\underbrace{a_1, \dots, a_1}_{s_1 \text{ times}}, \underbrace{a_2, \dots, a_2}_{(s_2 - s_1) \text{ times}}, \dots, \underbrace{a_j, \dots, a_j}_{(s_j - s_{j-1}) \text{ times}}, \dots).$$

The latter notation also will be used for general sequences  $a$  of finite length.

For each partition  $u$  one has a Schur functor  $\mathcal{S}_u : \mathcal{V} \rightarrow \mathcal{V}$ , where  $\mathcal{V}$  is the category of complex vector spaces and linear maps. If  $u_r > 0$  for some  $r > \dim V$ , one has  $\mathcal{S}_u V = 0$ .

By functoriality  $\mathcal{S}_u$  also operates on vector bundles over a given manifold  $X$ .

A generalized partition of length  $r$  is a weakly decreasing sequence  $u : I(r) \rightarrow \mathbb{Z}$ .

We define the diagram of a generalized partition  $u$  by

$$\mathcal{D}(u) = \{(i, j) \in I(r) \times \mathbb{Z} \mid j \leq u_i\}.$$

Note that this diagram is infinite even when  $u$  is a usual partition.

**Example**

For the generalized partition  $u = (2, 1, 0, -1, -2)$ ,  $\mathcal{D}(u)$  is the set of the following marked points :

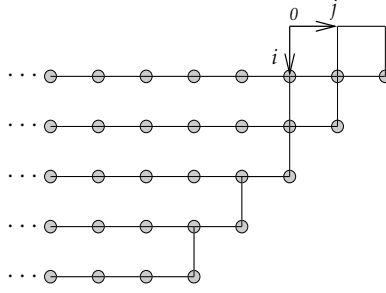


FIGURE 1. diagram of a partition

We define the involution

$$\chi^* : I(r) \times \mathbb{Z} \rightarrow I(r) \times \mathbb{Z} \text{ by } \chi^*(i, j) = (r + 1 - i, 1 - j),$$

and the reversed generalized partition  $\chi(u)$  by

$$\mathcal{D}(\chi(u))^c = \chi^*(\mathcal{D}(u))$$

where  $(\ )^c$  denotes the complement in  $I(r) \times \mathbb{Z}$ . Explicitly

$$\chi(u) = (-u_r, \dots, -u_2, -u_1) \text{ for } u = (u_1, u_2, \dots, u_r).$$

For the category  $\mathcal{V}_r$  of complex vector spaces of dimension  $r$  we extend the Schur functor notation  $\mathcal{V}_r \rightarrow \mathcal{V}$  to generalized partitions of length  $r$  by

$$\mathcal{S}_{u-\mathbf{k}_r} V = \mathcal{S}_u V \otimes (\det V^*)^k, \quad k \in \mathbb{N}^*,$$

where  $\mathbf{k}_r$  is the partition  $(k, k, \dots, k)$  of length  $r$  and  $V$  is a complex vector space of dimension  $r$ .

For  $u = (u_1, u_2, \dots, u_r)$  we have

$$\mathcal{S}_u V^* \simeq \mathcal{S}_{\chi(u)} V.$$

If a sequence  $u$  is not weakly decreasing, we put  $\mathcal{S}_u V = 0$ .



If  $\mathcal{D}(v) \subset \mathcal{D}(u)$ , we say that the pair  $u, v$  forms a skew partition  $u/v$ . We define the diagram of such a skew partition by

$$\mathcal{D}(u/v) = \mathcal{D}(u) / \mathcal{D}(v)$$

where  $\mathcal{D}(u) / \mathcal{D}(v)$  denotes difference of sets,  
and the weight by

$$|u/v| = \text{card}(\mathcal{D}(u/v))$$

The reversed skew partition is given by

$$\chi(u/v) = \chi(v)/\chi(u)$$

### 3.2. On the dominance partial order and ampleness.

**Definition 3.3.** Let  $I = (i_1, i_2 \dots)$ ,  $J = (j_1, j_2 \dots)$  be partitions of the same weight. We say that

$$I \succeq J \quad \text{if} \quad \sum_{k=1}^l i_k \geq \sum_{k=1}^l j_k \quad \text{for any } l.$$

This relation is called the dominance partial order.

**Definition 3.4.** For partitions  $I, J$  of arbitrary weight, this definition is generalized in [LN] by  $I \preceq J$  for  $|J|I \preceq |I|J$ . Here the multiplication of a partition  $I$  by  $n \in \mathbb{N}$  is defined by  $n(i_1, i_2, \dots) = (ni_1, ni_2, \dots)$ .

#### Example

For the partition of weight 5  $I = (1, 1, 1, 1, 1)$ , and  $J = (2, 1)$  of weight 3, we have  $I \preceq J$  because  $1/5 < 2/3$ ,  $2/5 < 1$ ,  $3/5 < 1$  and  $4/5 < 1$ .

The following lemma of Macdonald concern partitions of the same weight, but his proof can be adapted easily to partitions of arbitrary weight.

**Lemma 3.5.** [M, p.7]

*For any non-trivial partitions of arbitrary weight  $I, J$*

$$I \preceq J \iff \tilde{I} \succeq \tilde{J}.$$

In [LN] we also proved

**Lemma 3.6.** [LN]

*For any partitions  $I$  and  $J$ :*

$$\text{if } I \succeq J, \text{ then } \mathcal{S}_I E \text{ ample} \implies \mathcal{S}_J E \text{ ample.}$$

In particular,

$$\text{if } I \simeq J \text{ then } \mathcal{S}_I E \text{ ample} \iff \mathcal{S}_J E \text{ ample.}$$

We write  $I \simeq J$  if  $I \succeq J$  and  $I \preceq J$ . For example  $(k, 0, 0, \dots) \simeq (1, 0, 0, \dots)$

#### 4. ON THE LITTLEWOOD-RICHARDSON RULES

**Definition 4.1.** On any skew partition  $w/u$ , we define on  $\mathcal{D}(w/u)$  the Littlewood-Richardson (LR) order by

$$\begin{aligned} (i, j) <_{LR} (i', j') & \text{ for } i < i' \\ (i, j) <_{LR} (i, j') & \text{ for } j > j'. \end{aligned}$$

In this section  $w, u$  are fixed and  $x <_{LR} y$  implies  $x, y \in \mathcal{D}(w/u)$ .

Let  $w/u$  be a skew partition, and  $v = (v_1, \dots, v_r)$  a partition such that  $|v| = |w/u|$ . We denote  $c_1$  the numbering

$$c_1 : \mathcal{D}(w/u) \rightarrow \mathbb{N}^* \text{ with } \text{card}\{x \in \mathcal{D}(w/u) \mid c_1(x) = k\} = v_k, \forall k \in \mathbb{N}^*.$$

**Definition 4.2.** The numbering  $c_1$  is said to satisfy the LR rules, iff

- ( $L_1$ ) :  $c_1$  is strictly increasing on each column of  $\mathcal{D}(w/u)$ ,
- ( $L_2$ ) :  $c_1$  is weakly increasing on each row of  $\mathcal{D}(w/u)$ ,
- ( $L_3$ ) : for all  $x \in \mathcal{D}(w/u)$  and all  $k \in \mathbb{N}^*$ , one has  $\sigma_k(x) \geq \sigma_{k+1}(x)$ ,

where

$$\sigma_k(x) = \text{card}\{y \leq_{LR} x \mid c_1(y) = k\}$$

**Definition 4.3.** For a given  $c_1$ , let  $c_2 : \mathcal{D}(w/u) \rightarrow \mathbb{N}^*$  with  $c_2(x) = \text{card}\{y \leq_{LR} x \mid c_1(y) = c_1(x)\}$ .

Then  $c = (c_1, c_2)$ ,  $c : \mathcal{D}(w/u) \xrightarrow{\sim} Y(v)$  is a bijection. Let  $b = c^{-1}$ , with  $b = (b_1, b_2)$ .

**Lemma 4.4.** The map  $b$  is a strictly increasing function of the width on each row of  $Y(v)$  with respect to the order  $<_{LR}$ .

*Proof:*

Let  $x = (i, j)$ ,  $y = (i, j')$ ,  $j > j'$ . Then  $\text{card}\{z \leq_{LR} x \mid c_1(z) = i\} = j > j' = \text{card}\{z \leq_{LR} y \mid c_1(z) = i\}$ , thus  $b(i, j) >_{LR} b(i, j')$ .  $\square$

We put  $C(x) = \{c(y) \mid y \leq_{LR} x\}$ .

**Remark 4.5.**  $(i, j) \in C(x) \iff b(i, j) \leq_{LR} x$ . By the previous lemma,  $i, j \in \mathbb{N}^*$ ,  $(i, j) \in C(x) \iff j \leq \text{card}\{y \leq_{LR} x \mid c_1(y) = i\} = \sigma_i(x)$ .

**Lemma 4.6.** Assume that  $c_1$  satisfies the LR conditions for each  $x \in \mathcal{D}(w/u)$ . Then all  $C(x)$  are Young diagrams.

*Proof:*

We have to show  $(i, j) \in C(x) \implies (i', j) \in C(x)$  for  $i' = 1, \dots, i-1$ , and  $(i, j) \in C(x) \implies (i, j') \in C(x)$  for  $j' = 1, \dots, j-1$ . The first implication follows from

$j \leq \sigma_i(x) \leq^{(L_3)} \sigma_{i'}(x)$ . The superscript over the inequality sign indicates that this inequality holds by virtue of the corresponding property.

The second implication follows from remark 4.5.  $\square$

**Definition 4.7.** We say that the set  $C(x)$  satisfies the condition (Y), If for each  $x \in \mathcal{D}(w/u)$ ,  $C(x)$  is a Young diagram.

**Lemma 4.8.** Let  $c = (c_1, c'_2)$ ,  $c : \mathcal{D}(w/u) \longrightarrow Y(v)$  be a bijection, such that (Y) is true and  $c_1$  satisfies  $(L_1), (L_2)$ . Then  $c_1$  satisfies the LR rules and  $c'_2 = c_2$ .

*Proof:*

The first part of the conclusion is obvious, since  $C(x)$  is a Young diagram, and the length of the  $k$ -th row of  $C(x)$  is equal to  $\sigma_k(x)$ .

The restriction of  $c'_2$  to the set

$$\{x \in \mathcal{D}(w/u) \mid c_1(x) = i\}$$

is an order-preserving map to the  $i$ -th row of  $Y(v)$ , where the latter is ordered by the width, Indeed

For  $x, y \in \mathcal{D}(w/u)$  with  $c_1(x) = c_1(y) = i$ , we have  $x = b(c(x)) = b(i, c'_2(x))$ , and  $y = b(i, c'_2(y))$ , then  $c'_2(x) < c'_2(y) \iff x <_{LR} y$  by the Lemma 4.7.

Since both sets have  $v_i$  elements, this map is the unique order-preserving bijection.  $\square$

From now on we will use the

**Definition 4.9.** We say that a bijection  $c : \mathcal{D}(w/u) \xrightarrow{\sim} Y(v)$  satisfies the LR rules, iff  $(L_1), (L_2)$  of the definition 4.2 and (Y) are satisfied.

Recall that  $b = c^{-1}$ .

The importance of the LR rules is due to the following well-known proposition.

**Proposition 4.10.** Let  $\dim V = r$ ,  $u$  a generalized partition of length  $r$  and  $u'$  a partition. One has

$$\mathcal{S}_u V \otimes \mathcal{S}_{u'} V \simeq \bigoplus_{\substack{(w,b) \in LR(u,u') \\ 10}} \mathcal{S}_w V,$$

where  $LR(u, u')$  consists of pairs  $(w, b)$  such that  $w/u$  is a skew partition and

$$b : Y(u') \xrightarrow{\sim} \mathcal{D}(w/u)$$

a bijection satisfying the Littlewood-Richardson rules.

The LR rules have a useful symmetry which is hidden in their original definition, but will be made explicit in the following proposition.

**Proposition 4.11.**  *$c$  satisfies the LR rules, iff*

- (h): *On each column of  $\mathcal{D}(w/u)$ ,  $c_1$  preserves the order of the heights*
- (th): *On each column of  $\mathcal{D}(w/u)$ ,  $c_2$  weakly inverts this order*
- (w): *On each row of  $\mathcal{D}(w/u)$ ,  $c_2$  inverts the order of the widths*
- (tw): *On each row of  $\mathcal{D}(w/u)$ ,  $c_1$  weakly preserves this order*
- (h'): *On each column of  $Y(v)$ ,  $b_1$  preserves the order of the heights*
- (th'): *On each column of  $Y(v)$ ,  $b_2$  weakly inverts this order*
- (w'): *On each row of  $Y(v)$ ,  $b_2$  inverts the order of the widths*
- (tw'): *On each row of  $Y(v)$ ,  $b_1$  weakly preserves this order.*

*Proof:*

Conditions (h), (tw) are restatements of  $(L_1)$ ,  $(L_2)$ , and (tw') follows from lemma 4.4.

To show  $(L_1, L_2, Y) \implies (th)$  we show  $(L_1, L_2, L_3) \implies (th)$ , which is equivalent by the lemmas 4.6 and 4.8.

Let  $x, y, x', y' \in \mathcal{D}(w/u)$  with  $y = (i-1, j)$ ,  $x = (i, j)$

and  $y' = (i-1, j-1)$ ,  $x' = (i, j-1)$ , with  $c_1(y) = k$ ,  $c_1(x) = k'$ , by  $(L_1)$ ,  $k < k'$ . We have  $\sigma_k(x) \leq \sigma_k(y)$ .

Consider the case

a)  $\sigma_k(x) = \sigma_k(y)$

This case corresponds to  $y' \notin \mathcal{D}(w/u)$  or  $c_1(y') < c_1(x) - 1$ . We have by  $(L_3)$   $\sigma_{k'}(x) \leq \sigma_k(x)$  hence  $\sigma_{k'}(x) \leq \sigma_k(y)$ , which is by definition  $c_2(x) \leq c_2(y)$ .

b)  $\sigma_k(x) > \sigma_k(y)$

This case corresponds to  $y' \in \mathcal{D}(w/u)$  and  $c_1(y') \geq c_1(x) - 1$ .

Moreover

$$c_1(y') <^{(h)} c_1(x') \leq^{(tw)} c_1(x) \quad \text{and} \quad c_1(y') \leq^{(tw)} c_1(y) <^{(h)} c_1(x).$$

This implies

$$c_1(x) = c_1(x') \quad \text{and} \quad c_1(y') = c_1(y) = c_1(x) - 1.$$

This gives  $\sigma_{k'}(x) = \sigma_{k'}(x') - 1$  and  $\sigma_k(y) = \sigma_k(y') - 1$ .

Now we use the induction on  $j$ , so that we can assume  $c_2(x') \leq c_2(y')$ .

Thus

$$c_2(x) = \sigma_{k'}(x) = c_2(x') - 1 \quad \text{and} \quad c_2(y) = \sigma_k(y) = c_2(y') - 1.$$

The starting step of the induction is when  $x$  and  $y$  are such that  $y' \notin \mathcal{D}(w/u)$ , which is the case of a).

To prove (tw), (Y)  $\implies$  (w), assume  $(i, j), (i, j+1) \in \mathcal{D}(w/u)$ . We want to show that the assumption  $c_2(i, j+1) \geq c_2(i, j)$  leads to a contradiction.

Now  $c_1(i, j+1) \geq^{(tw)} c_1(i, j)$ , and the two inequalities together imply that  $c(i, j)$  belongs to the Young diagram  $C((i, j+1))$ , which is wrong, since  $(i, j) >_{LR} (i, j+1)$ .

To prove (h'), assume that  $i' < i$ ,  $b_1(i', j) \geq b_1(i, j)$ .

The case  $b_1(i', j) = b_1(i, j)$  is excluded by (w), which just has been proved. For  $b_1(i', j) > b_1(i, j)$ , we have  $(i', j) \notin C(b(i, j))$ ,  $(i, j) \in C(b(i, j))$ , which contradicts (Y).

To prove (h'), (w), (th)  $\implies$  (th') assume for  $x = (i, j), x' = (i', j)$ ,  $x, x' \in Y(v)$  that  $i' < i$ ,  $b_2(x) > b_2(x')$ . Let  $z = (b_1(x), b_2(x'))$ . Since  $b_1(x) > b_1(x')$  by (h') we have  $z \in \mathcal{D}(w/u)$ . This yields

$$j = c_2(b(x)) \stackrel{(w)}{<} c_2(z) \stackrel{(th)}{\leq} c_2(b(x')) = j, \text{ which is absurd.}$$

To prove (h), (tw), (Y)  $\implies$  (w') let  $(i, j'), (i, j) \in Y(v), j' < j$ . By lemma 4.4 we have  $b_1(i, j') \leq b_1(i, j)$ .

In the case  $b_1(i, j') = b_1(i, j)$ , we have  $b_2(i, j) < b_2(i, j')$ , by lemma 4.4, thus (w'). To prove (w') in the case  $b_1(i, j') < b_1(i, j)$ , assume  $b_2(i, j') \leq b_2(i, j)$ . Then  $z \in \mathcal{D}(w/u)$ , where  $z = (b_1(i, j), b_2(i, j'))$ .

$$\text{Thus } i = c_1(b(i, j')) \stackrel{(h)}{<} c_1(z) \stackrel{(tw)}{\leq} c_1(b(i, j)) = i, \text{ which is absurd.}$$

Finally let us show (h'), (w'), (tw')  $\implies$  (Y). The implication

$$(i, j) \in C(x), j' < j \implies (i, j') \in C(x) \text{ is equivalent to } (j' < j \implies b(i, j') <_{LR} b(i, j)).$$

For  $b_1(i, j') = b_1(i, j)$  this follows from (w') (or from (w).) The case  $b_1(i, j) <_{LR} b_1(i, j')$  is excluded by (tw').

The implication

$$(i, j) \in C(x), i' < i \implies (i', j) \in C(x) \text{ is equivalent to } (i' < i \implies b(i', j) <_{LR} b(i, j)).$$

By (h') one even has  $b_1(i', j) < b_1(i, j)$ .

□

**Remark 4.12.** Clearly the conditions (h, w, th, tw, h', w', th', tw') are not independent. For example, we have proved  $(h, tw, w', h', tw') \implies (h, tw, Y) \implies (w, th, th')$ .

**Remark 4.13.** The set of conditions remains invariant under the replacement

$$Y(v) \rightarrow Y(\tilde{v}), \quad \text{and} \quad \mathcal{D}(w/u) \rightarrow \chi^*(\widetilde{\mathcal{D}(w/u)}),$$

which exchanges columns and rows.

**Definition 4.14.** For  $U \subseteq \mathbb{Z} \times \mathbb{Z}$ , a map

$b : U \rightarrow \mathbb{Z} \times \mathbb{Z}$  with  $b(i, j) = (b_1(i, j), b_2(i, j))$  is called height increasing if  $b_1(i, j) \geq i$ ,  $\forall (i, j) \in U$

**Remark 4.15.** For partitions  $u, v$  of the same weight, the dominance partial order can be characterized by the property that  $u \preceq v$  if there is a height increasing bijection  $b : Y(v) \rightarrow Y(u)$ .

**Lemma 4.16.** If  $A \subset \mathbb{N}^* \times \mathbb{N}^*$  and  $b : A \rightarrow \mathbb{N}^* \times \mathbb{N}^*$  preserves the order of the height on each column of  $A$ , then  $b$  is height increasing.

*Proof:*

By induction, since  $1 \leq b_1(1, j) < b_1(2, j) < \dots < b_1(i, j)$  for all  $(i, j) \in A$ . This implies  $b_1(2, j) \geq 2$  etc ...

## 5. COHOMOLOGY GROUPS ON FLAG MANIFOLDS

For  $V$  a vector space of dimension  $d$  and a sequence  $s = (s_0, s_1, \dots, s_l)$  such that  $0 = s_0 < s_1 < s_2 < \dots < s_l < d$ , the flag manifold  $\mathcal{F}l_s(V) = \mathcal{F}l_{s_1, s_2, \dots, s_l}(V)$  given by subspaces  $V_{s_i} \subset V$  of codimension  $s_i$  has natural vector bundles  $Q_i$  with fibers  $V_{s_{i-1}}/V_{s_i}$ .

For a partition  $a = (a_1, a_2, \dots, a_l)$  such that  $a_1 > a_2 > \dots > a_l$ , we consider the Schur type line bundle

$$Q^a = \bigotimes_{k=1}^l \det(Q_k)^{a_k}.$$

Our aim is to prove

**Theorem 5.1.**

$$H^{p,q}(\mathcal{F}_s(V), Q^a) = \begin{cases} \delta_{q,0} \mathcal{S}_{a_s} V & \text{if } p = 0 \\ \bigoplus_{i \in I(p,q,s,a)} \mathcal{S}_{\rho(i)} V & \text{if } p \neq 0, \end{cases}$$

where  $|\rho(i)| = |a_s|$  and for all  $i$  in the index set  $I(p, q, s, a)$ ,

(i):

$$\rho(i) \prec a_s$$

(ii):

$$p + q + 1 + \|a_s\|^2 \leq \|\rho(i)\|^2.$$

The main tool for deriving such results is

**Bott's Theorem :** [ B ] or [D2]

Let  $V$  be a complex vector space of dimension  $d$  and  $\mathcal{F}(V)$  the complete flag manifold of  $V$ . Let  $a \in \mathbb{Z}^d$  and  $I(d) = (1, 2, \dots, d)$ .

Define  $\psi(a) = (a - I(d))^\geq + I(d)$ , where  $(a - I(d))^\geq$  is the sequence obtained by rearranging the terms of  $(a - I(d))$  in weakly decreasing order.

We call  $i(a)$  the number of strict inversions of  $(a - I(d))$ :

$$i(a) = \text{card}\{(i, j) \mid i < j, (a - I(d))_i < (a - I(d))_j\}.$$

Then

$$H^q(\mathcal{F}(V), Q^a) = \delta_{q, i(a)} \mathcal{S}_{\psi(a)} V.$$

Recall that one puts  $\mathcal{S}_{\psi(a)} V = 0$  if  $\psi(a)$  is not a partition. In particular the cohomology of  $Q^a$  is non vanishing iff all components of  $a - I(d)$  are pairwise distinct.

**Definition 5.2.** For  $a \in \mathbb{Z}^d$ , we say that  $a$  is admissible iff all components of  $a - I(d)$  are pairwise distinct.

**Corollary 5.3.** For the incomplete flag  $\mathcal{F}l_s(V)$ , we have [M1]

$$H^q(\mathcal{F}l_s(V), Q^a) = \delta_{q, i(a)} \mathcal{S}_{\psi(a_s)} V.$$

Let  $G_r(V)$  be the Grassmannian manifold of codimension  $r$  subspaces of  $V$ ,  $Q$  and  $S$  the universal quotient bundle and the universal subbundle on this manifold. Then the following statement holds:

**Corollary 5.4.** For  $u, v$  generalized partitions of lengths  $r$  and  $d - r$  respectively, we have [M1]

$$H^q(G_r(V), \mathcal{S}_u Q \otimes \mathcal{S}_v S) = \begin{cases} \mathcal{S}_{\psi(u, v)} V & \text{if } q = i(u, v), \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $|\psi(u, v)| = |u| + |v|$ .

Let  $w$  be a generalized partition of length  $r$  and  $u$  a partition, let  $a = (w, \tilde{u})$ , then the set of elements in  $(a - I(d))$  is  $\{\{\alpha_i\}_{i=1, \dots, r}, \{\beta_j\}_{j=1, \dots, d-r}\}$ , where  $\alpha_i = w_i - i$ ,  $\beta_j = \tilde{u}_j - (r + j)$ .

$(w, \tilde{u})$  is admissible iff  $\forall (i, j) \in I(r) \times I(d - r), \alpha_i \neq \beta_j$ . We have  $i(a) = \text{card}\{(i, j) \mid \alpha_i < \beta_j\}$ .

Let  $[\gamma]$  be the sequence such that,  $[\gamma]_i = \text{card}\{j \mid \alpha_i - \beta_j < 0\}$ , and  $\langle \gamma \rangle$  the sequence such that,  $\langle \gamma \rangle_j = \text{card}\{i \mid \alpha_i - \beta_j < 0\}$ .

**Lemma 5.5.** *Let  $w$  be a generalized partition of length  $r$  and  $u$  a partition, such that  $(w, \tilde{u})$  is admissible. Define  $s_+ : I(r) \rightarrow \mathbb{Z}$  by  $(s_+)_i = w_i + [\gamma]_i$  and  $s_- : \mathbb{N}^* \rightarrow \mathbb{Z}$  by  $(s_-)_j = \tilde{u}_j - \langle \gamma \rangle_j$ .*

*Then  $s_+, s_-$  are partitions and*

$$H^q(G_r(V), S_w Q \otimes \wedge_u S) = \delta_{q, i(w, \tilde{u})} \mathcal{S}_{\psi(w, \tilde{u})} V,$$

with

- (i):  $\psi(w, \tilde{u}) = (s_+, s_-)^\geq$ ,
- (ii):  $i(w, \tilde{u}) = |s_+| - \sum_{i \in I(r)} w_i$ .

*Proof:*

Since the cohomology group is given by Corollary 5.4, we only need to investigate the combinatorics.

Let  $a = (w, \tilde{u})$  be admissible. With the above notations the set of elements in  $(a - I(d))$  is  $\{\{\alpha_i\}_{i=1, \dots, r}, \{\beta_j\}_{j=1, \dots, d-r}\}$ , where  $\alpha_i$  is in position  $i$  in  $(a - I(d))$  and in position  $i + [\gamma]_i$  in  $(a - I(d))^\geq$ . Thus the term in this position in  $(a - I(d))^\geq + I(d)$  is  $\alpha_i + i + [\gamma]_i = (s_+)_i$ .

Similarly,  $\beta_j$  is in the position  $r + j$  in  $a - I(d)$ , and in the position  $r + j - \langle \gamma \rangle_j$  in  $(a - I(d))^\geq$ , such that the term in this position in  $(a - I(d))^\geq + I(d)$  is  $\beta_j + r + j - \langle \gamma \rangle_j = (s_-)_j$ .

For admissible  $a$  the sequences  $s_+$  and  $s_-$  are weakly decreasing. Indeed,

$$(s_+)_i - (s_+)_{i+1} = w_i - w_{i+1} + \text{card}\{j \mid \alpha_i > \beta_j > \alpha_{i+1}\} \text{ and}$$

$$\text{card}\{j \in \mathbb{N}^* \mid w_i - i > \beta_j > w_{i+1} - (i+1)\} \leq w_i - w_{i+1},$$

since  $\beta_j$  is a strictly decreasing integral sequence, and similarly for  $s_-$ .

Since  $s_-$  converges to 0, it is a partition. The sequence  $s_+$  has a finite number of terms, which are greater than the limit of  $s_-$ . Thus  $s_+$  is a partition, too.

Finally from  $(s_+)_i = w_i + [\gamma]_i$  we get (ii).  $\square$

For  $(w, \tilde{u})$  admissible, we will apply this lemma in the case where  $w/\chi(u)$  is a skew partition of length  $r$ .

In this case we denote by

$$\begin{aligned} \Sigma_\pm &= \{(i, j) \in \mathcal{D}(w/\chi(u)) \mid \beta_{1-j} \geq \alpha_i\}. \\ &= \{(i, j) \in \mathcal{D}(w/\chi(u)) \mid \tilde{u}_{1-j} - (r+1-j) \geq w_i - i\} \end{aligned}$$

Note that  $(w, \tilde{u})$  is admissible, iff  $\mathcal{D}(w/\chi(u)) = \Sigma_+ \cup \Sigma_-$ . We denote  $\mathcal{D}(w/\chi(u))$  by  $\Sigma$ .

**Definition 5.6.** We call  $(w/\chi(u))$  admissible iff  $\Sigma = \Sigma_+ \cup \Sigma_-$ , and we call  $\Sigma_+, \Sigma_-$  the splitting of  $\mathcal{D}(w/\chi(u))$



Recall that we put  $\tilde{u}_j = +\infty$  for  $j \leq 0$ , such that all  $(i, j) \in \Sigma$  with  $j > 0$  belong to  $\Sigma_+$ .

**Lemma 5.7.** *For  $(w, \tilde{u})$  admissible and  $w/\chi(u)$  a skew partition of length  $r$ , we have*

- (i):  $s_+ = [\Sigma_+]$
- (ii):  $s_- = \langle \chi^*(\Sigma_-) \rangle$
- (iii):  $i(w, \tilde{u}) = |u| - \text{card}(\Sigma_-)$ .

*Proof:*

$$\begin{aligned} (s_+)_i &= w_i + \text{card} \{j \leq 0 \mid \tilde{u}_{1-j} - 1 + j - r > w_i - i\} \\ &= w_i + \text{card} \{j \leq 0 \mid \beta_{1-j} > \alpha_i\}. \end{aligned}$$

If  $j \leq -u_{r+1-i}$  one has  $1 - j > u_{r+1-i}$ , thus  $\tilde{u}_{1-j} < r + 1 - i$ . Moreover since  $\mathcal{D}(\chi(u)) \subseteq \mathcal{D}(w)$ , we have  $-u_{r+1-i} \leq w_i$ . This implies  $j \leq w_i$ , thus  $\beta_{1-j} < \alpha_i$ , and we can write

$$(s_+)_i = w_i + \text{card} \{j \leq 0 \mid -u_{r+1-i} < j, \beta_{1-j} > \alpha_i\}.$$

If  $0 \geq j > w_i$ , we have  $1 - j \leq u_{r+1-i}$ . Thus  $\tilde{u}_{1-j} \geq r + 1 - i$  and  $\beta_{1-j} > \alpha_i$ .

This yields for any  $w_i \in \mathbb{Z}$

$$(s_+)_i = \text{card} \{j \in \mathbb{Z} \mid -u_{r+1-i} < j \leq w_i, \beta_{1-j} > \alpha_i\} = [\Sigma_+]_i.$$

as required.

Similarly we have

$$(s_-)_j = \tilde{u}_j - \text{card} \{i \in I(r) \mid \beta_j > \alpha_{r+1-i}\} = \tilde{u}_j - \langle \gamma \rangle_j.$$

a) If  $u_i < j$  we have  $\tilde{u}_j < i$ , and since  $\mathcal{D}(\chi(u)) \subseteq \mathcal{D}(w)$ , we have  $-u_i \leq w_{r+1-i}$ . this implies  $\beta_j < \alpha_{r+1-i}$ , thus

$$\langle \gamma \rangle_j = \text{card} \{i \in I(r) \mid -u_i < 1 - j, \beta_j > \alpha_{r+1-i}\}.$$

b) If  $j \leq -u_{r+1-i}$ , we have  $j < u_i$ , thus  $\tilde{u}_j \geq i$  and  $\alpha_{r+1-i} < \beta_j$ .

Now

$$\begin{aligned} \Sigma_- &= \{i \in I(r) \mid -u_{r+1-i} < j \leq w_i, \alpha_i > \beta_{1-j}\} \\ \chi^*(\Sigma_-) &= \{i \in I(r) \mid -u_i < 1 - j \leq w_{r+1-i}, \alpha_{r+1-i} > \beta_j\} \\ \langle \chi^*\Sigma_- \rangle_j &= \text{card} \{i \in I(r) \mid -u_i < 1 - j \leq w_{r+1-i}, \alpha_{r+1-i} > \beta_j\} \\ &= \text{card} \{i \in I(r) \mid -u_i < 1 - j, \alpha_{r+1-i} > \beta_j\}. \end{aligned}$$

The last equality is due to b). Now

$$\begin{aligned}
\langle \chi^* \Sigma_- \rangle_j + \langle \gamma \rangle_j &= \text{card} \{i \in I(r) \mid -u_i < 1-j, \alpha_{r+1-i} > \beta_j\} \\
&+ \text{card} \{i \in I(r) \mid -u_i < 1-j, \alpha_{r+1-i} < \beta_j\} \\
&= \text{card} \{i \in I(r) \mid -u_i < 1-j\} \\
&= \text{card} \{i \in I(r) \mid i \leq \tilde{u}_j\} = \tilde{u}_j
\end{aligned}$$

as required

Finally  $\text{card}(\Sigma_+) + \text{card}(\Sigma_-) = \sum_{i \in I(r)} (w_i - \chi(u)_i)$ . Together with  $\text{card}(\Sigma_+) = |s_+|$  and the previous lemma, this yields the desired formula for  $i(w, \tilde{u})$ .  $\square$

**Lemma 5.8.** *We have*

- (i):  $\psi(w, \tilde{u}) = ([\Sigma_+], \langle \chi^*(\Sigma_-) \rangle)^\geq$
- (ii):  $\langle \Sigma_- \rangle_j \leq [\Sigma_+]_i$  for  $(i, j) \in \Sigma_-$
- (iii):  $[\Sigma_+]_i \leq \langle \Sigma_- \rangle_j$  for  $(i, j) \in \Sigma_+$

*Proof:*

The preceding lemmas gives (i). We prove (ii), the proof of (iii) is similar.

Since  $[\Sigma_+]_i$  is decreasing, it suffices to consider for given  $j$

$$(i, j) \in \Sigma_-, (i+1, j) \notin \Sigma_-.$$

Determine  $j'$  such that  $(i, j') \in \Sigma_-, (i, j'+1) \notin \Sigma_-$ .

Since  $(r+1-\tilde{u}_{1-j}, j)$  is the element of smallest height in column  $j$  of  $\Sigma_-$  and  $(i, j)$  the element of greatest height, we have

$$\langle \Sigma_- \rangle_j = \tilde{u}_{1-j} - r + i \leq \tilde{u}_{1-j'} - r + i \leq w_i - j' = [\Sigma_+]_i.$$

$\square$

Since every row of  $\mathcal{D}(w, \chi(u))$  contains exactly one row of  $\Sigma_+$  and every column of  $u$  yields exactly one column of  $\Sigma_-$ , the length of  $\psi(w, \tilde{u})$  is the number of non-vanishing terms of  $[\mathcal{D}(w/\chi(u))]$  plus  $u_1$ .

**Example**

$$w = (5, 4, 3, 2, -1, -2)$$

$$u = (7, 7, 4, 3, 3, 1), \chi(u) = (-1, -3, -3, -4, -7, -7),$$

$$\tilde{u} = (6, 5, 5, 3, 2, 2, 2)$$

$$\alpha = (4, 2, 0, -2, -6, -8), \beta = (-1, -3, -4, -7, -9, -10, -11)$$

$$[\gamma] = (0, 0, 0, 0, 3, 4), \langle \gamma \rangle = (3, 2, 2, 1, 0, 0, 0)$$

$$s_+ = [\Sigma_+] = (5, 4, 3, 3, 2, 2), s_- = \langle \chi^* \Sigma_- \rangle = (3, 3, 2, 2, 2, 2).$$

The whole shaded area is  $w/\chi(u)$ , the set of vertical strips is  $\langle \chi^* \Sigma_- \rangle$ , the set of horizontal strips is  $[\Sigma_+]$

**Lemma 5.9.** *Choosing the permutation of shortest length for the re-ordering of  $([\Sigma_+], \langle \chi^*(\Sigma_-) \rangle)$  yields a natural bijection*

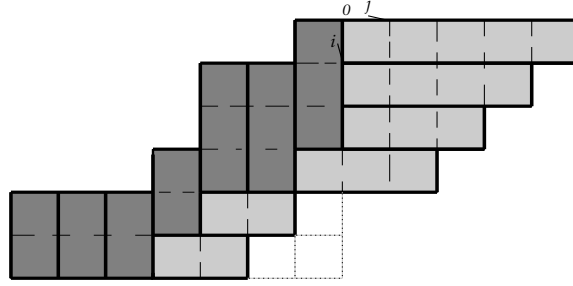


FIGURE 2.

$$\beta : \Sigma_+ \cup \Sigma_- \xrightarrow{\sim} Y(\psi(w, \tilde{u})).$$

Then  $\beta$  is height increasing.

*Proof:*

On  $\Sigma_+$ , the bijection  $\beta$  is height increasing, since  $[\Sigma_+]$  is weakly decreasing. For  $(i, j) \in \Sigma_-$ , the inequality derived above implies that the reordering puts  $\langle \Sigma_- \rangle_j$  after  $[\Sigma_+]_1, \dots, [\Sigma_+]_i$ .

Thus  $\beta_1(i, j) > i$ .

□

**Lemma 5.10.** *Let  $u, v$  be partitions such that  $l(u) \leq r$  and  $l(v) = r$ ,  $(w, b) \in LR(\chi(u), v)$ ,  $(w, \tilde{u})$  admissible, and  $\rho = \psi(w, \tilde{u})$ . Then either*

- 1)  $u = 0$ ,  $\rho = v$ , or
- 2)  $|u| + i(w, \tilde{u}) + 1 + \|v\|^2 \leq \|\rho\|^2$ .

*Proof:*

We use induction on  $u_1 + r$ , in other words on the length of  $\psi(w, \tilde{u})$ . For  $u_1 = 0$ , the statement is obviously true.

Assume  $u_1 > 0$ . We use the splitting of  $\mathcal{D}(w/\chi(u))$  into  $\Sigma_+, \Sigma_-$  introduced above.

Since  $v_r > 0$ , we either have  $\chi^*(1, u_1) \in \Sigma_-$  or  $\chi^*(1, u_1) \in \Sigma_+$ . The two cases will be treated differently.

**a)** For  $\chi^*(1, u_1) \in \Sigma_-$ , consider the partitions  $u', v'$  given by

$\mathcal{D}(u') = \mathcal{D}(u) \cap (I(r) \times u_1)^c$ , and  $Y(v') = b^{-1}(\mathcal{D}(w/\chi(u')))$ , in other words if  $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{u_1})$ , with  $\tilde{u}_{u_1} = l$ , then  $\tilde{u}' = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{u_1-1})$ . We denote by  $x_j = (r + 1 - j, 1 - u_1)$ ,  $j = 1, 2, \dots, l$  and  $L = \{x_j, j = 1, 2, \dots, l\}$

Let  $b'$  be the restriction of  $b$  to  $Y(v')$ . By Remark 4.13,  $Y(v')$  is a Young diagram. The Littlewood-Richardson rules yield  $(w, b') \in LR(\chi(u'), v')$ . Note that  $u'_1 = u_1 - 1$  and  $i(w, \tilde{u}) = i(w, \tilde{u}')$ . The skew

partition  $w/\chi(u')$  is admissible and the splitting of  $\mathcal{D}(w/\chi(u'))$  is given by  $\Sigma'_+ = \Sigma_+$ , and  $\Sigma_- = \Sigma'_- \cup L$

By Lemma 4.5 and Remark 4.13,  $Y(v)$  is obtained from  $Y(v')$  by successive unions with the preimages of the set  $L$ , each union being a Young diagram. Thus we have

$$||v||^2 - ||v'||^2 = \sum_{j=1}^L (2c_1(x_j) - 1),$$

Since  $b$  is height increasing ie  $c_1(x_j) \leq r + 1 - j$  this yields

$$||v||^2 - ||v'||^2 \leq l(2r - l).$$

The length of  $\rho = \psi(w, \tilde{u})$  is  $u_1 + r$ . With  $\rho' = \psi(w, \tilde{u}')$ , lemma 5.9 yields  $\rho = (\rho', l)$  and

$$||\rho||^2 - ||\rho'||^2 = l(2(u_1 + r) - 1).$$

By Lemma 5.8, we have  $l = \langle \Sigma_- \rangle_{-u_1} \leq [\Sigma_+]_r$ .

Since  $[\Sigma_+]_r > 0$  we have  $v'_r > 0$  and can use the induction assumption. Thus

$$\begin{aligned} |u| + i(w, \tilde{u}) + 1 + ||v||^2 - ||\rho||^2 &\leq \\ l + l(2r - l) - l(2(u_1 + r) - 1) + 1 & \\ = l(2 - l - 2u_1) + 1 &\leq 0, \end{aligned}$$

as required.

**b)** For  $\chi^*(1, u_1) \in \Sigma_+$ , the argument is similar. We put

$$\mathcal{D}(w'/\chi(u')) = \mathcal{D}(w/\chi(u)) \cap (\{r\} \times \mathbb{Z})^c, \text{ and } Y(v') = b^{-1}\mathcal{D}(w/\chi(u)),$$

in other words if  $u = (u_1, u_2, u_3 \dots)$  then  $u' = (u_2, u_3 \dots)$ .

Let  $b'$  be the restriction of  $b$  to  $Y(v')$ . The new partitions  $v', w'$  have length  $r' = r - 1$ . The skew partition  $w'/\chi(u')$  is admissible and the splitting of  $\mathcal{D}(w'/\chi(u'))$  is given by

$\Sigma'_- = \Sigma_-$ . Let  $[\Sigma_+]_r = l$ . We find

$$||v||^2 - ||v'||^2 \leq l(2r - 1),$$

$\rho = (\rho', l)$  and

$$||\rho||^2 - ||\rho'||^2 = l(2(u_1 + r) - 1).$$

Since  $\langle \Sigma_- \rangle_{1-u_1} \neq 0$ , we have  $u'_1 = u_1$  and  $v'_{r'} > 0$ , we can use the induction assumption. Altogether

$$\begin{aligned}
|u| + i(w, \tilde{u}) + 1 + ||v||^2 - ||\rho||^2 &\leq \\
2l + l(2r - 1) - l(2(u_1 + r) - 1) &= 2l(1 - u_1) \leq 0.
\end{aligned}$$

□

**Example of the situation of Lemma 5.10 case a)**

For  $u = (7, 7, 4, 3, 3, 1)$  and  $v = (9, 8, 6, 5, 5, 3)$ . We have  $w/\chi(u)$  on the left hand side the same as the previous example with the same  $[\Sigma_+]$ , and  $\langle \chi^* \Sigma_- \rangle$ . The shaded area is  $\tilde{u}_{u_1}$ . The partition  $v'$  is the partition  $v$  without the crossing boxes.

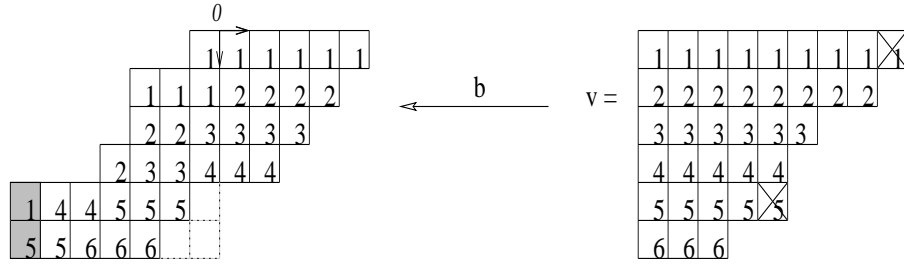


FIGURE 3.

**Lemma 5.11.** *Let  $v$  be a partition of length  $r$ . Then*

$$H^{p,q}(G_r(V), \mathcal{S}_v Q) = \bigoplus_{k \in K(p,q,r,v)} \mathcal{S}_{\rho(k)} V,$$

where  $|\rho(k)| = |v|$ . We have

- (i):  $\forall k \in K(p, q, r, v)$ ,  $\rho(k) \preceq v$ .
- (ii): Moreover,  $\rho(k) = v$  only occurs for  $p = q = 0$ , where

$$\text{for any partition } v \quad H^0(G_r(V), \mathcal{S}_v Q) = \mathcal{S}_v V.$$

*Proof:*

It is well known that

$$\Omega_{G_r(V)}^p = \wedge^p(Q^* \otimes S) = \bigoplus_{u \in \sigma^p} \mathcal{S}_u Q^* \otimes \mathcal{S}_{\tilde{u}} S,$$

where  $\sigma^p$  is the set of partitions of weight  $p$  and length  $r$ . Then

$$\begin{aligned}
H^{p,q}(G_r(V), \mathcal{S}_v Q) &= \bigoplus_{u \in \sigma^p} H^q(G_r(V), \mathcal{S}_{\chi(u)} Q \otimes \mathcal{S}_v Q \otimes \wedge_u S) \\
&= \bigoplus_{u \in \sigma^p} \bigoplus_{(w,b) \in LR(\chi(u), v)} H^q(G_r(V), \mathcal{S}_w Q \otimes \wedge_u S) \\
&= \bigoplus_{u \in \sigma^p} \bigoplus_{(w,b) \in LR(\chi(u), v)} \delta_{q, i(w, \tilde{u})} \mathcal{S}_{\psi(w, \tilde{u})} V.
\end{aligned}$$

For each term on the right-hand side. we have constructed a height increasing bijection

$$\beta \circ b : Y(v) \xrightarrow{\sim} Y(\psi(w, \tilde{u})).$$

Thus  $\psi(w, \tilde{u}) \preceq v$ . Since the length of  $\psi(w, \tilde{u})$  is at least  $u_1$  plus the length of  $v$ , it follows that  $\psi(w, \tilde{u}) = v$  implies  $u = 0$ , and thus  $p = 0$ .  $\square$

**Corollary 5.12.**  $H^{(p,q)}(G_r(V), \det Q) = 0$  if  $p \neq 0$  or  $q \neq 0$ ,

*Proof:*

there is no non-trivial partition strictly less than  $v = (1, 1, \dots, 1)$  of the same weight as  $v$ .

This is also a result of Le Potier [LP1, (corol.1)]  $\square$

In the sequel, we will use the notation  $R^{p,q} \mathcal{F}$  for  $R^q \pi_*(\Omega^p \otimes \mathcal{F})$ .

### 5.13. Proof of Theorem 5.1.

For  $p = 0$ , Corollary 1 to Bott's theorem gives  $i(a) = 0 = q$  and  $\psi(a_s) = a_s$ .

For  $p \neq 0$  we will use induction on  $l$ , the length of  $s$ . Let us consider the Borel-le Potier (B-L) spectral sequence associated to

$$\pi : Y = \mathcal{F}_s(V) \longrightarrow G_{s_l}(V) = X.$$

On  $X$  we have the canonical quotient bundle  $Q$  with fibres  $Q = V/V_{s_l}$ . The fibres of  $Y$  have the form  $\mathcal{F}l_{s'}(Q_x)$ , where  $s' = (s_1, \dots, s_{l-1})$ . On  $Y$  we have canonical bundles  $Q_i$  with fibres  $V_{s_{i-1}}/V_{s_i}$ .

As explained in the introduction, the Leray spectral sequence (L) associated to the projection  $\pi$ , called  ${}^{p',p} \mathcal{E}_L$  abuts to the  $\mathcal{E}_1$  terms of the Borel-Le Potier (BL) spectral sequence:

$${}^{p',p} \mathcal{E}_{2,L}^{q-j,j} \xrightarrow{L} {}^p \mathcal{E}_{1,B}^{p',q-p'} \xrightarrow{BL} H^{p,q}(\mathcal{F}_s(V), Q^a).$$

We have

$${}^{p',p} \mathcal{E}_{2,L}^{q-j,j} = H^{p',q-j}(G_{s_l}(V), R^{p-p',j} \pi_*(Q^a)).$$

On  $Q$  we have flags  $\{0\} \subset V_{s_{l-1}}/V_{s_l} \subset \dots V_{s_1}/V_{s_l} \subset V/V_{s_l}$ . For  $V_{s_j}/V_{s_l} = V'_{s_j}$  we have  $V'_{s_{j-1}}/V'_{s_j} = Q_j$ . We rewrite

$$\begin{aligned} Q^a &= (\det Q_1)^{a_1} \otimes \dots \otimes (\det Q_l)^{a_l} \\ &= (\det Q)^{a_l} \otimes (\det Q_1)^{a'_1} \otimes \dots \otimes (\det Q_{l-1})^{a'_{l-1}} \end{aligned}$$

where  $a'_i = a_i - a_l$ .

Setting  $a' = (a'_1, \dots, a'_{l-1})$ , we have

$$\begin{aligned} {}^{p',p}\mathcal{E}_{2,L}^{q-j,j} &= H^{p',q-j}(G_{s_l}(V), (\det Q)^{a_l} \otimes R^{p-p',j} \pi_*(Q^{a'})) \\ &= H^{p',q-j}(G_{s_l}(V), (\det Q)^{a_l} \otimes H^{p-p',j}(F_{s_1, s_2, \dots, s_{l-1}}(V), Q^{a'})). \end{aligned}$$

For  $p - p' = 0$ , the desired result follows from Corollary 5.4 with  $v = 0$  and Lemmas 5.10 and 5.11. For  $p' < p$  we have by the induction assumption for the graded bundle  $(Gr)$  associated to the higher direct image,

$$Gr R^{p-p',j} \pi_*(Q^{a'}) = \bigoplus_{k \in K} \mathcal{S}_{\rho'(k)} Q,$$

where  $\rho'(k) \preceq a'_{s'}$  for all  $k \in K$ . Consequently,

$$(\det Q)^{a_l} \otimes \mathcal{S}_{\rho'(k)} Q = \mathcal{S}_{\rho''(k)} Q \quad \text{with} \quad \rho''(k) \preceq a_s.$$

Since  $|\rho'(k)| = |a'_{s'}|$ , we have  $|\rho''(k)| = |a_s|$ .

The bundle  $R^{p-p',j} \pi_*(Q^{a'})$  on  $X$  is a homogeneous  $Gl(V)$ -bundle, thus specified by a representation of the stabilizer of a point in  $X$ . Since  $G(V_{s_l})$  acts trivially on the fibres, its representation factorizes through that of  $Gl(Q)$ . By the Schur lemma [FH], such a representation is reducible and is given by a Schur functor, which implies

$$R^{p-p',j} \pi_*(Q^{a'}) \simeq Gr R^{p-p',j} \pi_*(Q^{a'}).$$

Finally, again by the induction assumption,

$$p - p' + j + 1 + \|a'_{s'}\|^2 \leq \|\rho'(k)\|^2$$

for all  $k$  in the set of subscripts  $K$ . By Lemma 5.10 we have

$$p' + q - j + \|\rho'(k)\|^2 + |a_l|^2 \leq \|\rho(i)\|^2$$

for all  $i \in I(p, q, s, a)$ . Since  $\|a_s\|^2 = \|a'_{s'}\|^2 + |a_l|^2$ , the result follows from Lemmas 5.10 and 5.11.  $\square$

## 6. PROOF OF THEOREMS 1.1 AND COROLLARY 1.2

### 6.1. Proof of Theorem 1.1.

For a partition  $R = (r_1, \dots, r_m)$ , we take  $a = \tilde{R}^>$ ,  $s = (s_1, \dots, s_l) = R^<$ , such that  $a_s = \tilde{R}$ .

Let  $Y = \mathcal{F}_{s_1, \dots, s_l}(E)$  and  $\mathcal{L} = Q^a$ . We use induction on  $R$  with respect to the dominance partial order.

In the setup discussed in the introduction, we consider the Borel-Le Potier spectral sequence given by  $X, Y, \mathcal{L}, P$ , where  $\mathcal{L} = Q^a$ . Its  $\mathcal{E}_1$  terms can be evaluated by a Leray spectral sequence, for which

$${}^{p,P}\mathcal{E}_{2,L}^{q-j,j} = H^{p,q-j}(X, R^{P-p,j}\pi_*(Q^a)),$$

For  $P - p = 0$ , we have  $R^{0,j}\pi_*(Q^a) = \delta_{j,0} \wedge_r E$  by Theorem 5.1. This implies that for  $P - p = 0$  the Leray spectral sequence degenerates at  $\mathcal{E}_2$ , and  ${}^P\mathcal{E}_{1,B}^{P,q-P} = H^{P,q}(X, \wedge_r E)$ .

We have

$$\begin{array}{ccccccc} {}^P\mathcal{E}_{1,B}^{p-p_1,(q-1)-(p-p_1)} & \xrightarrow{d_{1,B}} & \dots & \xrightarrow{d_{1,B}} & {}^P\mathcal{E}_{1,B}^{p-1,(q-1)-(p-1)} & \xrightarrow{d_{1,B}} & {}^P\mathcal{E}_{1,B}^{p,q-p} \xrightarrow{d_{1,B}} 0 \\ & & & & \searrow d_{p_1,B} & & \\ & & \dots & & & & \end{array}$$

In order to show that  ${}^P\mathcal{E}_{1,B}^{p,q-P}$  is a subfactor of the limit group  $H^{P,q}(Y, Q^a)$ , we must prove that the sources of  $\xrightarrow{d_{p_1,B}} {}^P\mathcal{E}_{1,BL}^{p,q-p}$  vanish for each  $p_1 \neq 0$ .

Now each group  ${}^P\mathcal{E}_{1,B}^{p-p_1,(q-1)-(p-p_1)}$  is a subquotient of

$$H^{P-p_1,q-q_1-1}(X, R^{p_1,q_1}(Q^a)),$$

where by Theorem 5.1,

$$\mathrm{Gr} R^{p_1,q_1}\pi_*(Q^a) = \bigoplus \mathcal{S}_{\rho(i)}(E)$$

with  $\rho(i) \prec \tilde{R}$ , since  $p_1 \neq 0$ . By Lemmas 3.5 and 3.6 the vector bundles  $\mathcal{S}_{\rho(i)}(E)$  are ample. Moreover,  $p_1 + q_1 + 1 \leq |\rho|^2 - |R|^2$  by Theorem 5.1. Together with the assumption

$$P + q - n > \sum_{i=1}^m r_i(d - r_i) \quad (*)$$

and  $|\rho| = |R|$  this yields  $P - p_1 + q - q_1 - 1 - n > \sum_{i=1}^m \rho_i(d - \rho_i)$ . Since  $\rho(i) \prec \tilde{R}$  we can use the induction assumption, such that indeed  $H^{P-p_1,q-q_1-1}(X, \mathrm{Gr} R^{p_1,q_1}(Q^a))$  and consequently



$H^{P-p_1, q-q_1-1}(X, R^{p_1, q_1}(Q^a))$  and  ${}^P\mathcal{E}_{1,B}^{p-p_1, (q-1)-(p-p_1)}$  are zero.

It may well be that  $R^{p_1, q_1}(Q^a)$  is isomorphic to the associated graded bundle  $Gr R^{p_1, q_1}(Q^a)$ , as in the Grassmannian case discussed above. We did not investigate this question, since it is not necessary for our proof.

We have shown that under the condition  $(*)$   ${}^P\mathcal{E}_{1,B}^{P, q-P} = H^{P, q}(X, \wedge_r E)$  is a subquotient of  $H^{P, q}(Y, Q^a)$ .

Now  $Q^a$  is ample by

**Lemma 6.2.** *Let  $a = (a_1, a_2, \dots)$  a strictly decreasing partition, and  $s$  as above.*

*If  $\mathcal{S}_{a_s} E$  is ample over  $X$ , then  $Q^a$  is ample over  $Y$ .*

*Proof:* See [D1], Lemmas 2.11 and 4.1. □

To conclude the proof of Theorem 1.1, the group  $H^{P, q}(Y, Q^a)$  vanishes under the condition of Kodaira-Akizuki-Nakano theorem but the condition  $(*)$  implies this condition.

### 6.3. Proof of Corollary 1.2.

We need the following

**Lemma 6.4.**  $\otimes_{i=1}^l S^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E$  is ample  $\iff \mathcal{S}_{\tilde{\lambda}} E$  is ample, where  $\tilde{\lambda} = (s_1, s_2, \dots, s_m, \underbrace{1, \dots, 1}_{k_1 \text{ times}}, \underbrace{1, \dots, 1}_{k_2 \text{ times}}, \dots, \underbrace{1, \dots, 1}_{k_m \text{ times}})$ .

*Proof:*

$\mathcal{S}_{\tilde{\lambda}} E$  is direct summand of  $\otimes_{i=1}^l S^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E$ .

Conversely for any  $\mathcal{S}_{\lambda_i} E$  direct summand of  $\otimes_{i=1}^l S^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E$ , we have  $\lambda \succeq \lambda_i$ , then we use Lemma 3.5. □

Now

Among  $\mathcal{S}_{\lambda_i} E$  where  $\lambda_i = (r_1, r_2, \dots)$ , that appear as direct summands of  $\otimes_{i=1}^l S^{k_i} E \otimes_{j=1}^m \wedge^{s_j} E$ , the optimal condition  $\sum_{i=1}^m r_i(d - r_i)$  is obtained by  $\lambda$ .

This concludes the proof of Corollary 1.2 □

### REFERENCES

- [KAN] Y. Akizuki, S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc.Jap.Acad. **30** (1954), 266-272.
- [B] R.Bott, *Homogeneous vector bundles*, Ann. Math. **66** (1957), 203-248.
- [D1] J.P. Demailly, *Vanishing theorems for powers of an ample vector bundle*, Invent. Math. **91** (1988), 203-220.

- [D2] B. Demazure, *A very simple proof of Bott's theorem*, Invent. Math. **33** (1976), 271-220.
- [FH] W. Fulton, J. Harris *Representation theory*, Springer Verlag
- [G] P.A. Griffiths, *Hermitian differential geometry, Chern classes and positive vector bundles*, Global analysis, paper in honor of Kodaira (1969), 185-251.
- [KV] Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43-46.
- [L] F. Laytimi, *On Degeneracy Loci*, International Journal of Mathematics **6** (Vol 7) (1988), 203-220.
- [LN] F. Laytimi, W. Nahm *Vanishing theorems for product of exterior and symmetric powers*, e-print math.AG/9809064.
- [LP1] Le Potier, *Cohomologie de la Grassmannienne à valeurs dans les puissances extérieures et symétriques du fibré universel*, Math. Ann. **226** (1977), 257-270.
- [LP2] J. Le Potier, *Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque*, Math. Ann. **218** (1975), 35-53.
- [M] I.G. Macdonald, *Symmetric Functions and Hall polynomials*, Clarendon Press, Oxford (1979) (1997), 401-416.
- [M1] L. Manivel, *Un théorème d'annulation pour les puissances extérieures d'un fibré ample*, J. reine angew. Math. **422** (1991), 91-116.
- [M2] L. Manivel, *Vanishing theorems for ample vector bundles*, Invent.math. **127** (1997), 401-416.
- [S] A.J. Sommese, *Submanifold of Abelian Varieties*, Math. Ann. **233** (1978), 229-256.

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